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(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. SIXTH SEMESTER EXAMINATION, MAY 2016
THIRD YEAR [BATCH 2013-16]
MATHEMATICS (Honours)
Paper : VII

Date : 11/05/2016
Time : 11 am – 3 pm
Full Marks : 100

[Use a separate Answer Book for each group]

Group – A

Answer any three of the following : [3×10]

1. a) Define Gamma-function $\Gamma(n)$ and find a necessary and sufficient condition (on n) for its convergence. [1+4]

   b) Show that : $\displaystyle\int_0^\infty \frac{\sin ax \sin bx}{x}dx = \frac{1}{2}\log\left(\frac{a+b}{a-b}\right)$, $0 < b < a$. [5]

2. a) Examine the convergence of the following improper integrals : [3+2]

   i) $\displaystyle\int_0^1 \frac{x^p \log x}{1+x^2}dx \ \ (p \in \mathbb{R})$.

   ii) $\displaystyle\int_0^\infty \log(1+\operatorname{sech} x)dx$.

   b) Prove that $\displaystyle\int_0^\infty \frac{x}{1+x^4\cos^2 x}dx$ is divergent. [5]

3. a) Find the Fourier series of f, where $f(x) = x^2$ in $(-\pi, \pi]$ and $f(x+2\pi) = f(x)\ \forall x \in \mathbb{R}$. Hence show that $\displaystyle\frac{\pi^2}{6} = \sum_{n=1}^\infty \frac{1}{n^2}$. [4]

   b) Prove that $\displaystyle\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$. [3]

   c) Prove that : $\sqrt{\pi}\,\Gamma(2n) = 2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)$, $n > 0$. [3]

4. a) Let $f:[0,1]\times[0,1] \to \mathbb{R}$ be defined by

   $$f(x,y) = \begin{cases} 0; & x \text{ irrational} \\ 0; & x \text{ rational}, y \text{ irrational} \\ \frac{1}{q}; & x \text{ rational}, y = \frac{p}{q} \text{ in least terms} \end{cases}.$$

   Show that f is integrable and $\displaystyle\int_{[0,1]\times[0,1]} f = 0$. [5]

   b) If f is bounded and integrable on $[-\pi, \pi]$ and if a_n, b_n are its Fourier coefficients, then prove that $\displaystyle\operatorname*{Lt}_{n\to\infty} a_n = 0 = \operatorname*{Lt}_{n\to\infty} b_n$. [5]

5. a) By changing to polar coordinates, show that $\iint_E \sqrt{x^2+y^2}\,dx\,dy = \frac{38\pi}{3}$ where E is the region in the xy-plane bounded by $x^2+y^2=4$, $x^2+y^2=9$. [5]

b) Evaluate the volume of the sphere $x^2+y^2+z^2=r^2$ using multiple integral ($r > 0$). [5]

Answer <u>any two</u> of the following : [2×10]

6. a) Prove that if a and b are integers, with $b>0$, then there exist unique integers q and r satisfying $a=qb+r$, where $2b \le r < 3b$. [5]

b) State and prove the Chinese Remainder Theorem. [5]

7. a) If p is a prime number, then prove that $(p-1)! \equiv -1 \pmod{p}$. [4]

b) Prove that there are an infinite number of primes of the form 4n + 3. [3]

c) f and F be number-theoretic functions such that $F(n)=\sum_{d|n} f(d)$. Then prove that, for any positive integer N, $\sum_{n=1}^{N} F(n) = \sum_{k=1}^{N} f(k)\left[\frac{N}{k}\right]$. [3]

8. a) Define Möbius μ-function. Hence prove that for each positive integer $n \ge 1$,

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}$$

where d runs through the positive divisors of n. [4]

b) Show that $\phi(3n)=3\phi(n)$ if and only if 3|n, where n is a positive integer. [3]

c) Let n be a positive integer. Prove that $\tau(n)$ is odd if and only if n is a perfect square. [3]

<u>Group - B</u>

Answer <u>any three</u> of the following : [3×10]

9. a) If two numbers p and q are chosen at random from the set {1, 2, 3, ...,10} with replacement, determine the probability that the roots of the equation $x^2+px+q=0$ are real. [5]

b) A and B are two independent witnesses in a case. The probability that A will speak the truth is x and probability that B will speak the truth is y. A and B agree in a certain statement. Show that the probability that this statement is true is $\frac{xy}{1-x-y+2xy}$. [5]

10. a) If n coins are distributed among m beggars at random, what is the probability that a particular beggar will get exactly k coins? [4]

b) The probability density of a random variable X is

$$f(x) = 2xe^{-x}, \quad x>0$$
$$= 0, \quad \text{elsewhere}.$$

Find the probability density for X^2. [2]

c) Two numbers are chosen at random between 0 and 2. Find the probability that their sum of squares is less than 2. [4]

11. a) Two people agree to meet at a definite place between 12 and 1 O'clock with the understanding that each will wait 20 minutes for the other. Find the probability that they will meet. [5]

b) If X is a binomial (n, p) variate, then prove that $\mu_{K+1} = p(1-p)\left(nk\mu_{K-1} + \frac{d\mu_K}{dp}\right)$, where μ_K is the kth central moment. [5]

12. a) If $\rho(X,Y)$ denotes the correlation coefficient between the random variables X and Y then prove that $-1 \le \rho(X,Y) \le 1$. [4]

b) Find the C.F. of X, whose density function is

$$f(x) = \frac{1}{2}, \ |x| < 1$$
$$= 0, \text{ elsewhere}$$
. [4]

c) Show that $\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2\,\text{cov}(X,Y)$. [2]

13. a) Let X be a random variable with mean m_x and variance σ_x^2 (finite).Then prove that for each $\in > 0$, $P(|X - m_x| \ge \in) \le \frac{\sigma_x^2}{\in^2}$. [5]

b) A special unbaised die with 10 faces is thrown 1000 times. Find the lower bound of the probability that the face 10 occurs between 80 to 120 times. [5]

Answer <u>any two</u> of the following : [2×10]

14. a) Prove the following statement : Let $f(z) = u(x,y) + iv(x,y)$ where $(z = x + iy)$, u and v are real valued functions be defined on $D \subseteq \mathbb{C}$ except at z_0. Then $\lim_{z \to z_0} f(z) = \ell + i\ell'$ (where $\ell, \ell' \in \mathbb{R}$) iff $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = \ell$ & $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = \ell'$. (where $z_0 = x_0 + iy_0$). [4]

b) For $z \ne 0$ consider $f(z) = \frac{\bar{z}}{z}$. Check whether $\lim_{z\to 0} f(z)$ exists or not. [2]

c) Let $T \subseteq \mathbb{C}_\infty$. Then prove that the corresponding image of T on the Riemann Sphere S is—

i) circle in S not containing (0, 0, 1) if T is a circle.

ii) a circle in S minus (0, 0, 1) if T is a line. [4]

15. a) Test for uniform convergence of $\sum_{n=1}^{\infty} \frac{1}{n^2 + z^2}$ in the region $1 < |z| < 2$. [3]

b) Prove that the function f defined by $f(z) = \sqrt{|xy|}$ satisfy the C–R equation at origin but is not differentiable there. [2]

c) Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$, where Ω is open in $\mathbb{C}$ & $z_0 \in \mathbb{C}$.

Let $f(z) = u(x,y) + iv(x,y), (z = x + iy) \ \forall z \in \Omega$. State the necessary and sufficient condition for the differentiability of f at z_0 & prove it. [1+4]

16. a) Test the convergence of $\{z_n\}$ where $z_n = i\frac{(-1)^n}{n^2} - 2$. [2]

b) Find the radius of convergence of $\sum_{n=0}^{\infty} (n + 2^n) z^n$. [3]

c) Prove that a power series and its k times derivative power series both have the same radius of convergence. [5]

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